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Calculating the Present Value
of Riskless Cash Flows

by
Richard S. Ruback

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Abstract

This paper uses arbitrage arguments to show that the present value of riskless cash flows is determined by discounting them at the after-tax discount rate. Present value formulas are derived for riskless cash flows in three cases: (i) uniform interest and tax rates; (ii) certain, but not uniform interest and tax rates; and (iii) uncertain interest rates. The relation between the after-tax discount rate approach and the adjusted present value approach is also explored.

Calculating the Present Value of Riskless Cash Flows

Richard S. Ruback*

1. Introduction

This note analyzes a fundamental valuation question:

Is it appropriate for a firm to calculate the present value of riskless after-tax cash flows by discounting at the riskless interest rate or at the after-corporate-tax riskless interest rate?

I asked numerous colleagues this question. Each of them thought the answer was obvious. However, approximately half of them thought the obvious answer was the before tax interest rate, whereas the other half believed the answer to be the after-tax interest rate. Introductory finance textbooks also seem to disagree on the answer. While the textbooks do not contain an answer to this specific question, Brealey and Myers (1981) suggest an adjusted present value approach in which the cash flow is valued as the sum of two components: the cash flows discounted at the before-tax interest rate and the realized tax

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shields discounted at the before-tax interest rate. Weston and Brigham (1981) imply that the cash flows should be discounted at the after-tax riskless rate. Neither Brealey and Myers (1981) nor Weston and Brigham (1981) provide a rigorous demonstration of why their methods are appropriate.

In this note I use arbitrage arguments to prove that the present value of a stream of after-tax riskless cash flows is determined by discounting them at the after-tax discount rate. This result is, of course, not novel. However most other papers, such as Henderson (1976), reach this conclusion by applying the tax corrected cost of capital formula of Modigliani and Miller (1963).¹ The tax corrected cost of capital is not applicable to any arbitrary stream of cash flows and is based on various restrictive assumptions that limit the dependence between financing decisions and other corporate policies such as investment decisions. In contrast, the arbitrage approach used in this paper involves no such restrictions. Furthermore, the arbitrage approach provides valuation formulas when interest and tax rates are certain, but not uniform and when interest rates are uncertain.

In the analysis the firm constructs the arbitrage by issuing riskless debt with after-tax payments that exactly offset the stream of after-tax cash flows being valued. This offsetting debt is feasible since the firm can secure it with the riskless stream of cash inflows. Furthermore, the offsetting debt eliminates changes in the amount of net debt which would otherwise be associated with the project. Net riskless debt is defined as the present value of riskless cash outflows less the present value of riskless cash

1 Two papers, Myers, Dill and Bautista (1976) and Franks and Hodges (1978), show that leases should be valued using after-tax discount rates without relying on the tax corrected cost of capital formula. Myers, Dill and Bautista (1976) argue that leasing displaces debt and use dynamic programming to obtain their result. Franks and Hodges (1978) use arbitrage arguments similar to those used in this paper to obtain their lease evaluation formula.



inflows. Since the outflows of the offsetting debt exactly offset the riskless inflows from the project in each period, the arbitrage proof does not result in any changes in the firm's riskless net debt. Since there are no changes in the amount of net riskless debt, no specification of the complex interaction between investment and financing decisions is required.

The arbitrage proofs use the market price of riskless securities and therefore do not depend on the determinants of the riskless interest rate. The analysis does not imply that there is a corporate advantage to debt financing or that the project should be financed with debt. For example, Miller (1977) presents a model in which there are no personal taxes on income from common stocks and, in equilibrium, the after-corporate-tax interest rate on riskless bonds equals the required rate of return on riskless equity. In the context of the arbitrage proof, Miller's model implies that the present value of the stream of riskless cash flows is unaffected by constructing the portfolio of offsetting securities using riskless debt or riskless equity.

If there are corporate tax advantages to debt financing, the proceeds from a portfolio of offsetting securities constructed with riskless debt will be higher than an offsetting portfolio constructed with riskless equity. Alternatively, if there is a corporate tax disadvantage to debt financing, the after-tax interest rate will exceed the required rate of return on riskless equity.² The proceeds from an offsetting portfolio constructed from riskless equities will yield higher proceeds than a portfolio of offsetting debt.

The arbitrage proof formalizes the standard conceptual definition of the present value of a stream of cash flows: the present value of a stream of

² I am not aware of any theory which implies a tax disadvantage to debt financing.



riskless cash flows is the minimum amount of money that has to be invested in riskless securities to replicate the stream of cash flows. The proceeds from the offsetting securities, therefore, equals the present value of the riskless cash flows. In theory either riskless debt or riskless equity could be used to construct the portfolio of offsetting securities. The required rate of return on riskless equity is not, however, readily observable. In contrast, the riskless interest can easily be obtained from the prices of government bonds. Therefore, this paper focuses on portfolios of offsetting securities constructed by issuing riskless debt. To the extent that the firm realizes the value of the interest tax shields created by the offsetting debt, the opportunity cost of capital for the riskless cash flows is the after-tax riskless interest rate.

The arbitrage proofs using offsetting debt literally establishes a lower bond on the present value of the future riskless cash flows. The proof does not ensure that an offset riskless equity portfolio would not provide greater proceeds than the offsetting debt. However, this occurs only when there is a corporate tax disadvantage to borrowing so that the after-tax riskless interest rate exceeds the return on riskless equity. Furthermore, if issuing the offsetting equity is feasible and provides greater proceeds than the offsetting debt, the firm can construct an arbitrage between the portfolios.³ Such arbitrage opportunities are inconsistent with equilibrium security prices.

³ The arbitrage is constructed by buying riskless debt and selling riskless equity. In particular, the firm buys $\$M$ worth of riskless debt with payments of $\$X_t$ in period t . The firm sells equity claims to these riskless cash flows. Since the after-tax riskless rate exceeds the return on riskless equity, the proceeds from the sale of equity claims exceeds $\$M$ and the firm realizes an arbitrage profit.



2. The present value of a single riskless cash flow

In this section I show that, when interest and tax rates are constant and the firm realizes the full value of its interest tax shields, the present value of a single riskless after-tax cash flow is determined by discounting it at the after-tax riskless interest rate. When interest rates and marginal tax rates are certain, but not constant over time, the present value is calculated by discounting the riskless after-tax cash flow at after-tax interest rates that correspond to the term structure and the firm's marginal tax rate. When future one period interest rates are uncertain, the present value of riskless cash flows can be determined using the yields to maturity that are implicit in current pure discount bond prices. Finally, discounting the riskless cash flow at the after-tax riskless interest rate is equivalent to the adjusted present value technique recommended by Myers (1974) in which the present value of riskless cash flows is determined by discounting the after-tax cash flows at the before-tax riskless interest rate plus the interest tax shields on the offsetting debt discounted at the before-tax riskless interest rate.

2.1 Constant interest and tax rates

Proposition: If a firm realizes the full value of its interest tax shields, the present value of a riskless stream of after tax cash flows is determined by discounting them at the after-tax riskless rate of interest.

Proof: Suppose a firm is going to receive a riskless cash flow of X_T in T periods. The firm pays taxes equal to the corporate tax rate, τ , times



the cash flow so that the after-tax cash flow is $X_T(1 - \tau)$. To offset the riskless cash inflow with a riskless cash outflow the firm constructs the offsetting debt as follows: In the initial period the firm borrows B_0 . In the next period the firm repays $B_0(1 + r)$ to retire its debt. It finances this repayment from an interest tax shield of τB_0 and a new loan of $B_1 = B_0(1 + r(1 - \tau))$. This rollover process continues in each period prior to period T . That is, in period t the firm repays the face value of its debt from the prior period, $B_{t-1}(1 + r)$, with its interest tax shield τB_{t-1} and a new loan of $B_t = B_{t-1}(1 + r(1 - \tau))$.

The initial amount borrowed, B_0 , is determined so that the loan from period $T - 1$ is repaid in period T with the after-tax cash flow, $X_T(1 - \tau)$, and the tax shield τB_{T-1} . In period T ,

$$\begin{aligned} X_T(1 - \tau) &= B_{T-1}(1 + r) - \tau B_{T-1} \\ &= B_{T-1}(1 + r(1 - \tau)) . \end{aligned}$$

Since the loan is refinanced each period,

$$B_t = B_{t-1}(1 + r(1 - \tau))$$

and, by recursive substitution,

$$X_T(1 - \tau) = B_0(1 + r(1 - \tau))^T.$$

Since the only net cash flow from the riskless cash inflow of $X_T(1 - \tau)$ and the offsetting debt is the initial amount borrowed, B_0 , the present value of $X_T(1 - \tau)$ is,

$$PV(X_T(1 - \tau)) = B_0 = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T} . \quad (1)$$



In summary the present value of a single after-tax cash flow is obtained by discounting it at the after-tax riskless interest rate. It provides the present value of any after-tax riskless cash flows including depreciation tax shields and income subject to capital gains treatment. To apply the proposition to these cash flows, the numerator in (1), $X_T(1 - \tau)$, is replaced by the after-tax riskless cash flow being valued.

2.2 Extensions

2.2.1 Non-Uniform certain interest and tax rates: Changes in interest rates can be incorporated by using pure discount bond to offset the cash flows being valued. If future one period interest rates are certain, the offsetting debt is constructed by rolling over one period discount bonds. Similarly, if future tax rates are certain, the offsetting debt is constructed using the realized interest tax shields. In particular, the offsetting debt is constructed as follows: In the initial period the firm borrows B_0 . In the next period the firm repays the outstanding debt with its realized interest tax shield of $\tau_1 r_1 B_0$ and a new loan of

$$B_1 = B_0(1 + r_1(1 - \tau_1))$$

where τ_1 and r_1 are the tax and interest rates in period 1. This rollover process continues in each period t prior to period T . That is, in each period t , the firm repays the face value of its loan from the prior period, $B_{t-1}(1 + r_t)$, with the realized interest tax shield of $\tau_t r_t B_{t-1}$ and a new loan of $B_t = B_{t-1}(1 + r_t(1 - \tau_t))$. The face value of the offsetting debt in period T is $B_{T-1}(1 + r_T)$. By recursive substitution,

$$B_{T-1}(1 + r_T) = (1 + r_T) B_0 \prod_{t=1}^{T-1} (1 + r_t(1 - \tau_t)).$$



The initial amount borrowed, B_0 , is determined by equating the required repayment of the face value of the offsetting debt in period T to the after-tax riskless cashflow of $X_T(1 - \tau_T)$ plus the realized interest tax shield of $\tau_T r_T B_{T-1}$:

$$(1 + r_T) B_0 \prod_{t=1}^{T-1} (1 + r_t(1 - \tau_t)) = X_T(1 - \tau_T) + \tau_T r_T B_0 \prod_{t=1}^{T-1} (1 + r_t(1 - \tau_t))$$

or,

$$X_T(1 - \tau_T) = B_0 \prod_{t=1}^T (1 + r_t(1 - \tau_t)).$$

Since the only net cash flow from the riskless cash inflow of $X_T(1 - \tau_T)$ and the offsetting debt is the initial amount borrowed, B_0 , the present value of $X_T(1 - \tau_T)$ is:

$$PV(X_T(1 - \tau_T)) = B_0 = \frac{X_T(1 - \tau_T)}{\prod_{t=1}^T (1 + r_t(1 - \tau_t))} \quad (2)$$

2.2.2 Uncertain interest rates: If future one period interest rates are uncertain and the tax rates are certain, the firm can use the current prices of pure discount bonds to value the after-tax cash flow. The firm offsets the after-tax cash flow being valued by short selling pure discount bonds and the proceeds from these short sales is the present value the after-tax cash flow. The tax code allows the firm to deduct the yield on the implicit loan balance in each period. That is, in period t the firm is allowed to deduct interest of $B_0 R_T(1 + R_T)^{t-1}$ where B_0 is the initial amount borrowed and R_T is the yield on the pure discount bond that matures in period T .

Suppose the firm is going to receive a single riskless after-tax cash flow of $X_T(1 - \tau)$ in T periods and that the firm realizes the full value of interest tax shields. The tax rate is assumed to be constant and equal to



τ in each period.⁴ Define Z_T as the dollar amount of pure discount bonds that are sold short to offset the cash flow in period T. The face value of these bonds is $Z_T(1 + R_T)^T$ where R_T is the yield riskless pure discount bond which pays one dollar at time T. In period T the firm receives the riskless after-tax cash flow from the project, $X_T(1 - \tau)$, and the interest tax shield from the pure discount bond that matures in period T $\tau Z_T R_T(1 + R_T)^{T-1}$. In period T, the firm must pay $Z_T(1 + R_T)^T$ to cover its short sale. The dollar amount of pure discount bonds sold short to offset the cash flow that occurs in time T, Z_T , is determined by equating the inflows and outflows in period T:

$$Z_T(1 + R_T)^T = X_T(1 - \tau) + \tau Z_T R_T(1 + R_T)^{T-1}.$$

Rearrangement yields,

$$Z_T = \frac{X_T(1 - \tau)}{(1 + R_T)^{T-1}(1 + R_T(1 - \tau))}$$

The pure discount bond that matures in period T creates tax shields in each period. The firm offsets these intermediate tax shields by short selling pure discount bonds that mature in the period the tax shield is realized. For example, in period T-1 the firm receives an interest tax shield of $\tau Z_T R_T(1 + R_T)^{T-2}$ from the pure discount bonds that mature in period T and an interest tax shield of $\tau Z_{T-1} R_{T-1}(1 + R_{T-1})^{T-2}$ from the

⁴ The arbitrage technique cannot be used to value cash flows when tax rates are uncertain because traded securities which enable the firm to hedge the tax rate uncertainty do not exist.



bonds that mature at T-1. The amount of bonds short sold to offset the cash flow in period T-1, Z_{T-1} , is determined by equating inflows and outflows:

$$Z_{T-1}(1 + R_{T-1})^{T-1} = \tau Z_T R_T (1 + R_T)^{T-2} + \tau Z_{T-1} R_{T-1} (1 + R_{T-1})^{T-2}$$

Rearrangement yields:

$$Z_{T-1} = \frac{\tau Z_T R_T (1 + R_T)^{T-2}}{(1 + R_{T-1})^{T-2}(1 + R_{T-1}(1 - \tau))} \quad (3)$$

The bonds that mature in periods T and T-1 provide tax shields in period T-2 which are offset using a discount bond that matures in T-2. The amount of bonds short sold to offset the period T-2 tax shields are:

$$Z_{T-2} = \frac{\tau Z_T R_T (1 + R_T)^{T-3} + \tau Z_{T-1} R_{T-1} (1 + R_{T-1})^{T-3}}{(1 + R_{T-2})^{T-3}(1 + R_{T-2}(1 - \tau))} \quad (4)$$

The general expression for the amount of pure discount bonds that are short sold to offset the tax shields that are realized in period T-K is:

$$Z_{T-K} = \frac{\tau \sum_{t=0}^{K-1} Z_{T-t} R_T (1 + R_T)^{T-t-1}}{(1 + R_{T-K})^{T-K-1}(1 + R_{T-K}(1 - \tau))} \quad \text{for } K = 1, 2, \dots, T-1 \quad (5)$$

The offsetting debt constructed by borrowing Z_t dollars of pure discount bonds that mature in time t is feasible since the firm can secure the loans with the riskless cash flow from the project and the riskless tax shields. Also, while future one period interest rates are uncertain, the current prices of pure discount bonds and their associated yields to maturity are known. Finally, the offsetting debt offsets all cash flows associated with the project so that there is no change in the amount of net riskless debt.

The proceeds from the offsetting debt created by the short sales is:



$$PV(X_T(1 - \tau)) = \sum_{t=1}^T Z_t \quad (6)$$

and equals the net present value of the project. Note that in the absence of constant interest rates the present value of a riskless after-tax cash flow is not determined by discounting at the after-tax yield to maturity on a T-period pure discount bond. This occurs because the intermediate tax shields from the offsetting debt are discounted at yields relevant to the periods in which the tax shields are realized. Expression (6) does, however, reduce to (1) if the interest rates are constant.⁵

⁵ When the term structure is uniform, the yield on discount bonds of any maturity equals the one period rate, r . Equation (3) becomes:

$$Z_T = \frac{X_T(1 - \tau)}{(1 + r)^{T-1}(1 + r(1 - \tau))}$$

The general expression for Z_{T-K} reduces to:

$$Z_{T-K} = \frac{\tau \sum_{t=0}^{K-1} Z_{T-K}}{1 + r(1 - \tau)}$$

The present value of the cash flow equals the sum of the Z_t . Note that

$$Z_T + Z_{T-1} = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]$$

Similarly,

$$Z_T + Z_{T-1} + Z_{T-2} = \frac{1 + r}{1 + r(1 - \tau)} (Z_T + Z_{T-1}) = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^2$$

This recursive substitution implies that

$$\sum_{t=1}^T Z_t = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^{T-1}$$

Substituting the expression for Z_T yields

$$\sum_{t=1}^T Z_t = \frac{X_T(1 - \tau)}{(1 + r)^{T-1}(1 + r(1 - \tau))} \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^{T-1} = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T}$$

2.3 The Adjusted Present Value Approach

When interest rates are uncertain, the proceeds from short selling discount bonds to offset a single riskless after-tax cash flow, (6), can be expressed as:

$$PV(X_T(1-\tau)) = \frac{X_T(1-\tau)}{(1+R_T)^T} + \tau \sum_{t=1}^T \frac{\sum_{K=t}^T Z_K R_K (1+R_K)^{t-1}}{(1+R_t)^t} \quad (7)$$

Expression (7) shows that the present value of a single riskless after-tax cash flow can be calculated as the sum of the after-tax cash flow discounted at the before-tax yield (the first term on the right hand side) plus the interest tax shields in each period discounted at the corresponding before-tax yield (the second term on the right hand side). This formulation is, therefore, similar to the adjusted present value approach recommended by Myers (1974).

Expression (7) shows that the tax shields from the offsetting debt depend on the value of the riskless after-tax cash flow. For example, when the term structure is uniform, equation (7) simplifies to:

$$PV(X_T(1-\tau)) = V_0^T = \frac{X_T(1-\tau)}{(1+r)^T} + \sum_{t=1}^T \frac{\tau r V_t^T}{(1+r)^t} \quad (8)$$

where V_t^T is the present value of $X_T(1-\tau)$ in period t . The second term on the right hand side of (8) is the interest tax shields discounted at the before tax interest rate. Note that the balance of the offsetting debt equals the present value of the riskless after-tax cash flow in each period.

To apply the adjusted present value approach the present value of the after-tax cash flow is required to determine the appropriate tax shields.

Applications of the adjusted present value technique which determine the balance of the offsetting debt, or equivalently the interest tax shields, from the cost of project which generates the riskless after-tax cash flow or from the after-tax cash flow discounted at the before-tax interest rate are, therefore, incorrect. Also according to equation (1), the value of the riskless cash flow grows at the after-tax discount rate,

$$V_t^T = V_{t-1}^T (1 + r(1 - \tau)).$$

Since the balance of the offsetting debt equals the present value of the riskless after-tax cash flow, the balance of the offsetting debt increases at the after-tax discount rate. Other patterns of the offsetting will not generate the appropriate tax shields. For example, the interest tax shields from an equal repayment schedule will not generate the correct tax shields because the balance of such an annuity declines over time instead of increasing.

Pragmatically, when interest rates are certain determining the present value of a riskless after-tax cash flow by discounting it at the after-tax interest rate using (1) or (2) appears to be much easier than the adjusted present value approach (8) since the present value of the cash flow (including the associated interest tax shields) is required to determine the appropriate tax shields to include in the adjusted present value formulation. However, when interest rates are uncertain, discounting at the after-tax yields using (6) appears to be as difficult as the adjusted present value approach (7) since both approaches require construction of the offsetting portfolio of pure discount bonds.

2.4 Valuing Streams of Cash Flows

The valuation formulas presented in section 2 apply to a single riskless cash flow. These formulas can be easily extended to value a sequence of riskless cash flows. Since present values are additive, the present value of a sequence of cash flows is the sum of the present values of the individual cash flows in the sequence. In other words, the appropriate valuation formula is used to value each cash flow and the sum of these values equals the net present value of the sequence of cash flows.

3. Conclusions

This paper uses arbitrage arguments to prove that the present value of a stream of riskless after-tax cash flows is determined by discounting them at the after-tax riskless interest rate. In the proof, the firm constructs an equivalent loan such that the loan payments exactly offset the stream of riskless after-tax cash flows. The proceeds from this equivalent loan equal the minimal present value of the riskless after-tax cash flows. Since the firm realizes interest tax shields from this equivalent loan, the appropriate discount rate to value the stream of riskless after-tax cash flows is the after-tax riskless rate.

It is worthwhile to examine the argument for using the before-tax interest rate to value riskless after-tax cash flows in light of the proof that the after-tax discount rate is appropriate. Arguments for using before-tax discount rate typically focus on the investors' opportunity cost of capital instead of the corporation's cost of capital. For example, investors can invest in riskless bonds directly. The proceeds from this investment, before personal taxes, is the riskless interest rate. Thus the investors' opportunity cost of capital for riskless investments is the before-corporate

tax riskless rate. The after-tax earnings on riskless corporate investment must, therefore, equal or exceed the before-tax riskless rate to compensate investors for their opportunity cost of capital.

The argument for using the before-tax interest rate to value riskless after-tax cash flows ignores the corporation's ability to generate interest tax shields by borrowing against the riskless cash flows. Since this borrowing offsets the riskless after-tax cash flow, the appropriate discount rate for riskless after-tax cash flows is the corporation's cost of capital for riskless borrowing which is the after-tax riskless interest rate.

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